# Misiurewicz points in one-dimensional quadratic maps

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Misiurewicz points are constituted by the set of unstable or repellent points, sometimes called *the set of exceptional points*. These points, which are preperiodic and eventually periodic, play an important role in the ordering of hyperbolic components of one-dimensional quadratic maps. In this work we use graphic tools to analyse these points, by measuring their preperiods and periods, and by ordering and classifying them.

## 1. Introduction

To study one dimensional quadratic maps, we propose to use the real axis neighbourhood (antenna) of the Mandelbrot-like set of the corresponding complex quadratic form, that offers graphic advantages. However, we must take into account that only the intersection of the complex form and the real axis have a sense in one-dimensional quadratic maps. All the one-dimensional quadratic maps are equivalent because they are topologically conjugate [1]. This means that any one-dimensional quadratic map can be used to study the others. We chose the map $x_{n+1} = x_n^2 + c$, due to the historical importance of its complex form, the Mandelbrot map $z_{n+1} = z_n^2 + c$ [2,3].

As is well known, the Mandelbrot set can be defined by

$$\mathrm{M} = \left\{ c \in \mathrm{C} : f_c^{\circ k}(0) \nrightarrow \infty \text{ as } k \to \infty \right\} \qquad (1)$$

where $f_c^{\circ k}(0)$ is the $k$-iteration of the $c$ parameter-dependent quadratic function $f_c(z) = z^2 + c$ ($z$ and $c$ complex) for the initial value $z_0 = 0$.

The Mandelbrot set is complex, so it has a real and an imaginary part. Therefore, $\mathsf{M} = \mathrm{Re}(\mathsf{M}) + \mathrm{Im}(\mathsf{M})$. Although we manage complex figures in the real axis neighbourhood (antenna) for our own convenience, the map $x_{n+1} = x_n^2 + c$ can be studied by looking at the real part of the set, namely $\mathrm{Re}(\mathsf{M})$, which is the intersection of $\mathsf{M}$ and the real axis.

Recently, we have used the antenna of the Mandelbrot set to study the ordering of hyperbolic components into a one-dimensional quadratic map [4]. For such an ordering different levels of *separators* were required. Separators were Misiurewicz points in all cases.

Misiurewicz points, which are preperiodic and eventually unstable periodic points, have been widely studied [5-9]. They take their name from the Polish mathematician Michal Misiurewicz [5] who became well known in the study of dynamics of one-dimensional maps. Unstable or repellent points, which belong to what is sometimes called *the set of exceptional points* [10], were well known early. Later, band-merging points [11] which belong to the set of exceptional points were determined. As is pointed out by Peitgen et al [8], Douady and Hubbard [6] proved that Misiurewicz points are repellent and are dense at the boundary of the Mandelbrot set. In the case of one-dimensional quadratic maps, based on the presumed denseness of the periodic intervals, many early workers conjectured that chaos occurrred only on a set of zero measure, but this is already known to be wrong (Ge et al [12]) after the work of Jakobson [13] and the numerical results by Farmer [14]. Tan Lei [15] proved the narrow relationship between Julia and Mandelbrot sets at the Misiurewicz points.

Preceding works have in most cases a mathematical approach. We try to do a complementary study of the Misiurewicz points. Since they have already been well defined, we now try to measure their preperiods and periods, to see their position, to order them and count their number, to see their physical sense if they have one, etc. In order to do so, we take a more experimental physical approach to the work by making many hundreds of measurements of these points, which was possible due to our graphic tools [16,17]. One-dimensional quadratic maps normaly are studied by means of its bifurcation diagram. We have used the real axis neighbourhood (antenna) of the Mandelbrot-like set of the corresponding complex quadratic map, that offers graphic advantages. In fact, a narrow window is replaced for a midget, and the filament of this midget can give a valuable information about the period of the midget when the Mandelbrot set is drawn by the escape line method [16].

We shall begin by an easy introduction about Misiurewicz points. Afterwards, we shall analyse Misiurewicz points which appear in a natural way when hyperbolic components are ordered into a one-dimensional quadratic map. They are the *characteristic Misiurewicz points* whose main property is that they have the same period as the chaotic band where they are found. Finally, as we shall see, there is an infinite number of Misiurewicz points whose periods are different from those of the chaotic band where they are located. These points are the *non-characteristic Misiurewicz points*.

## 2. Misiurewicz points

### *2.1. What is a Misiurewicz point?*

Let $x_{k+1} = x_k^2 + c$ be a one-dimensional quadratic map. This map has the critical point $x = 0$ and then, the initial value generating superstable orbits is $x_0 = 0$. A parameter value *c* is called a Misiurewicz point if the critical point is preperiodic, but non periodic itself, i.e., if *c* is a point that has to cover a preperiod (also called *tail* by Misiurewicz [5]) before reaching a periodic orbit. We name this point $\mathrm{M}_{n,p}$, where *n* is the preperiod and *p* the period. So, $\mathrm{M}_{n,p}$ is a Misiurewicz point *n*-preperiodic and eventually *p*-periodic.

For the map $x_{k+1} = x_k^2 + c$, let us turn the numerical iteration into the dynamical system of critical polinomials $\mathrm{P}_{n+1} = \mathrm{P}_n^2 + c$ with the initial value $\mathrm{P}_0 = 0$ (according Zeng et al [18]). Then

$$\mathrm{P}_0 = 0, \tag{2}$$

$$\mathrm{P}_1 = c,$$

$$\mathrm{P}_2 = c^2 + c,$$

$$\mathrm{P}_3 = (c^2 + c)^2 + c,$$

...

These polynomials are the dark lines seen in the usual bifurcation diagram [18-20]. Chaos bands merge where these dark lines cross each other, and periodic windows open where dark lines touch upper and lower edges.

In fig. 1 we show the orbit $\{x_0, x_1, x_2, x_3, x_4, x_5, x_6\}$ of a 4-preperiodic and eventually 3-periodic Misiurewicz point $\mathrm{M}_{4,3}$. The preperiod is $\{x_0, x_1, x_2, x_3\}$, and the period is $\{x_4, x_5, x_6\}$. We have $x_7 = x_4$, and the parameter value $c$ can be obtained resolving de equation $\mathrm{P}_7 = \mathrm{P}_4$.

Then, the parameter value of a Misiurewicz point $\mathrm{M}_{n,p}$ is calculated by the expresion $\mathrm{P}_{n+p} = \mathrm{P}_n$, and Misiurewicz points $\mathrm{M}_{n,p}$ are the intersection of critical polynomials $\mathrm{P}_{n+p} \cap \mathrm{P}_n$. Normally, there are several Misiurewicz points $\mathrm{M}_{n,p}$. When it is needed, we add a superindex in brackets to order the different Misiurewicz points with the same $n$ and $p$ from the Feigenbaum point to the end of the antenna. So, there are three $\mathrm{M}_{4,3}$: $\mathrm{M}_{4,3}^{(1)} = -1.790327491...$, $\mathrm{M}_{4,3}^{(2)} = -1.981056903...$, and $\mathrm{M}_{4,3}^{(3)} = -1.988429001...$ .

*2.2 Algebra of Misiurewicz points*

Let us see fig. 1 again. We observe that if after 4 iterations we reach a cycle of period 3, after 5 iterations we also reach the cycle of period 3. And it is the same after 6, 7, ... iterations. Likewise, we observe that if after 4 iterations we reach a cycle of period 3, we have also reached a cycle of period 6 (3·2), 9 (3·3) and successive multiple of 3.

Then we can give the following property: *If a Misiurewicz point is n-preperiodic and eventually p-periodic, it is also* $(n+q)$*-preperiodic and eventually* $(p \cdot r)$*-periodic, where* $q = 0,1,2,...$ *and* $r = 1,2,3,...$. Or, in other words

$$\mathrm{M}_{n,p} = \mathrm{M}_{n+q,p\cdot r}, \quad q = 0,1,2,..., \quad r = 1,2,3,... \tag{3}$$

All the infinite ways of being represented a Misiurewicz point form an equivalence class. When a Misiurewicz point is given, it is always referred to the representative of the equivalence class, which is obtained when $q = 0$ and $r = 1$.

*2.3. Counting the number of Misiurewicz points*

The combinatorial problem of enumerating all periodic sequences of a given length was solved by Gilbert and Riordan [21] using group theory and combinatorial techniques. Afterwards, Metropolis, Stein and Stein (MSS) [10] have tabulated numerical results for the number of periodic orbits of the U-sequence, and they have given a formula to calculate this number in the prime period case. Later, May [22] outlined a recursive procedure for the counting of stable *p*-cycles in unimodal iterations. More recently, Lutzky [23] has given a explicit formula for the number of primitive, stable *p*-cycles associated with unimodal iterations, and also has given expresions for the number of hiperbolic components, cardioids, and discs which are associated with *p*-cycles for the Mandelbrot set. Finally, Hao and Xie [24] and Xie and Hao [25] have studied the problem of the counting the number of periods in one-dimensional maps with multiple critical points.

We have undertaken the study of the number of Misiurewicz points $\mathrm{M}_{n,p}$ in one-dimensional quadratic maps [26]. In this work we give an algorithm to obtain the symbolic sequence of all possible Misiurewicz points $\mathrm{M}_{n,p}$ for a given *n* and *p*. The number of Misiurewicz points found by this algorithm has been checked numerically. In

Table I the number of Misiurewicz points for the values $0 \le n \le 8$ and $1 \le p \le 5$ is given. So, we have 39 Misiurewicz points $\mathrm{M}_{6,5}$ and 56 Misiurewicz points $\mathrm{M}_{8,4}$.

Table I
Number of Misiurewicz points $\mathrm{M}_{n,p}$ for the values $0 \le n \le 8$ and $1 \le p \le 5$.

| *p* | *n:* | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | 1 | 0 | 1 | 1 | 1 | 1 | 3 | 3 | 9 |
| 2 | | 1 | 0 | 0 | 1 | 2 | 3 | 4 | 9 | 14 |
| 3 | | 1 | 0 | 0 | 2 | 3 | 6 | 10 | 19 | 34 |
| 4 | | 2 | 0 | 0 | 4 | 6 | 8 | 18 | 30 | 56 |
| 5 | | 3 | 0 | 0 | 8 | 14 | 20 | 39 | 72 | 136 |

Of course, a superstable periodic point is simply a Misiurewicz point with preperiod zero [5]. So, in table I, there are 3 superstable points of period 5 (according with MSS [10]). Note that the number of Misiurewicz points of period 5 increases with the preperiod. Then, for a given small period, the number of Misiurewicz points with large preperiod is enormous.

*2.4. Misiurewicz points are unstable*

The real part of the Mandelbrot set that we are studying is defined for the parameter values $-2 \le c \le 1/4$. In this part of the real axis there are several kinds of periodic points according to the multiplier value. For the map $f(x) = x^2 + c$, let $x_0$ be a periodic point of period-*k*. Feigenbaum [27] defines the multiplier $\rho$ of the cycle as the derivative of $f^{\circ k}$ at $x_0$. Using the chain rule we see that

$$\rho = (f^{\circ k})'(x_0) = \prod_{j=0}^{k-1} f'(x_j) \qquad (4)$$

so that the derivative of $f^{\circ k}$ is the same at all points of the cycle.

If $|\rho|>1$, we have a repelling (unstable) cycle. This is the case of a Misiurewicz point. This can clearly be seen in fig. 2, where we show the graphical iteration [27] of $f(x)=x^2+c$ for the initial value $x_0=0$ when $c=-1.43035763245130...$ This parameter value correspond to the Misiurewicz point $\mathrm{M}_{5,2}^{(1)}$, which is the band-merging point from the period-4 chaotic band to the period-2 chaotic band. The fig. 2a has been obtained by iterating 20 times the function. The orbit enters the period-2 cycle after 5 iterations (dotted line), and the other 15 iterations remain in the period-2 cycle. If *c* were a stable periodic point, the orbit would remain in the cycle forever. But *c* is a Misiurewicz point, and hence an unstable point. Therefore, after several iterations, the orbit will leave the period-2 cycle. The fig. 2b has been obtained by iterating 120 times, and it can be seen how the orbit begins to leave the period-2 cycle. Finally, in fig. 2c which were obtained by iterating 5000 times, the orbit has completely left the period-2 cycle and has a chaotic behaviour. In addition, in fig. 2a we have drawn also the graph of the second iteration $f^{\circ 2}(x)$ to show the slopes in the fixed points A and B of the period-2 orbit (they have multiplier $|\rho|>1$).

*2.5. Symbolic sequences of Misiurewicz points*

According to Hao and Zheng [28], the symbolic dynamic has its origin in the theory of topological dynamical systems, and is used to rigorously describe the orbits of such systems. In its abstract mathematical form it is difficult to use. However, applied symbolic dynamics is becoming a more practical tool in the case of one-dimensional unimodal maps [10].

It is well know that a period-*p* orbit has a symbolic sequence (or pattern) with *p* letters (C's, L's and R's) properly combined. The meaning of these letters are: centre (C), left (L) and right (R) and they indicate the position of the point of the orbit when the graphical iteration is represented. This can be extended to the Misiurewicz points. Since we have not seen the symbolic sequence of a Misiurewicz point in any place, we shall represent the preperiod in brackets followed by the period without brackets. So, the symbolic sequence of $M_{n,p}$ is given by *n* letters in brackets followed by *p* letters without brackets. If we take the previous example $M_{5,2}^{(1)}$ from fig. 2, we deduce that its symbolic sequence is (CLRLL)LR or, what is the same, $(\mathrm{CLRL}^2)\mathrm{LR}$. In fact, since $x_0 = 0$, the first value is always in the centre of the parabola and is a C. The next are L and R. From here, the letters are different according to the cases. In our case they are LL with which the preperiod is finished, followed by a cycle LR of period-2.

In the case of hyperbolic components of maps like the logistic map $f(x) = \lambda x(1-x)$, Schroeder [20] and Zheng and Hao [29] gave rules to order same length symbolic sequences according to their parameter values. It is very easy to convert these rules to the case of the Mandelbrot real map. So, we say that the periodic orbit $P_a$ dominates the periodic orbit $P_b$ if the parameter absolute value of the first orbit $|c_a|$ is greater than the parameter absolute value of the second orbit $|c_b|$. The dominant symbolic sequence is obtained by seeing the principal strings (formed by the common leading string plus the first different letter) of the two symbolic sequences. The dominant symbolic sequence is that which principal string is odd (odd number of L's). We have experimentally seen that this rule can also be used to order symbolic sequences of the Misiurewicz points with the same length of preperiod and period.

We have extended the former rule to different length symbolic sequences of both hyperbolic components and Misiurewicz points in the case of the Mandelbrot real map. The new rule can be enunciated as follows: *If the principal strings have different parity, the dominant symbolic sequence is that with odd principal string. If the principal strings have the same parity, the dominant symbolic sequence is that with a greater number of R's if the principal strings are odd or that with a less number of R's if the principal strings are even.*

For instance, let us order three pairs of points (both superstable points or Misiurewicz points) of the Mandelbrot real map. Let $M_{6,5}^{(27)} = (\mathrm{CLR}^4)\mathrm{LRL}^3$ and $M_{4,4}^{(2)} = (\mathrm{CLR}^2)\mathrm{LRL}^2$ be the symbolic sequences of two different length Misiurewicz points [26]. Their principal strings are $\mathrm{CLR^2R}$ (odd) and $\mathrm{CLR^2L}$ (even). Therefore, $\mathrm{M}_{6,5}^{(27)}$ dominates $\mathrm{M}_{4,4}^{(2)}$. In fact, $\mathrm{M}_{6,5}^{(27)} = -1.989727200...$ and $\mathrm{M}_{4,4}^{(2)} = -1.826254739...$ Now, let $\mathrm{P}_5^{(3)} = \mathrm{CLR}^3$ and $\mathrm{P}_7^{(8)} = \mathrm{CLR^4L}$ be the symbolic sequences of two different length periodic superstable orbits [10]. Both principal strings ($\mathrm{CLR}^3$ and $\mathrm{CLR^3R}$) are odd. Therefore, the symbolic sequence with a greater number of R's, the second one, is the dominant. In fact, $\mathrm{P}_5^{(3)} = -1.985424253...$ and $\mathrm{P}_7^{(8)} = -1.991814172...$ To finish, let $\mathrm{M}_{7,5}^{(5)} = (\mathrm{CLR^2LRL})\mathrm{L^4R}$ and $\mathrm{M}_{4,1}^{(1)} = (\mathrm{CLR}^2)\mathrm{L}$ be two different length Misiurewicz points. Both principal strings are even, and the second symbolic sequence is the dominant because has a less number of R's. In fact $\mathrm{M}_{7,5}^{(5)} = -1.821213808...$ and $\mathrm{M}_{4,1}^{(1)} = -1.892910987...$

For readers accustomed to the logistic map $f(x) = \lambda x(1-x)$, we have to point out some considerations. Firstly, it is well know that $\lambda$-parameter values and corresponding $c$-parameter values of both (logistic and Mandelbrot) maps are related by $\lambda = 1 + \sqrt{1-4c}$. Therefore, the point $\mathrm{M}_{2,1}$ placed in $c = -2$ in the Mandelbrot real map, is placed in $\lambda = 4$ in the logistic map. Secondly, to find the symbolic sequence of a Misiurewicz point in the logistic map, we have to interchange L's and R's in the Mandelbrot real map sequence.

*2.6. Other properties of the Misiurewicz points*

Let us do a quick revision of other properties of Misiurewicz points that we can find in the literature about this topic. In the first place, we point out that the hyperbolic componentes and the Misiurewicz points form precisely the set of $c$-values of $\mathsf{M}$ for wich the orbit of 0 under $z^2 + c$ is finite. The hyperbolic components are contained in the interior of $\mathsf{M}$, and the Misiurewicz points are on the boundary of $\mathsf{M}$, since there exist arbitrarily small perturbations of $z^2 + c$ such that 0 escapes to $\infty$ [3].

Another important property of Misiurewicz points is that they are dense at the boundary of $\mathsf{M}$. The Misiurewicz points are located in a complicated pattern in the antenna area of $\mathsf{M}$ [30]. However, since Misiurewicz points are the zeros of a sequence of polynomials, they are countable [30].

Douady [31], tell us about external arguments of points in $\mathcal{M}$. Most of the remarkable points, in Julia sets as well as in the Mandelbrot set, have external arguments wich are rational. Centers of components have no external argument, since they are inside. The root of each component has two external arguments, wich are rational with odd denominator; each Misiurewicz point has one or several external arguments (one if it is the end of a filament, three or more if it is a branch point), wich are rational with even denominator.

If $c$ is a Misiurewicz point, the Julia set of this point is a dentrite, i.e. it has no interior [8] but it is locally connected.

Let $c$ be a Misiurewicz point. Tan Lei [15] has proved that the Julia set and the Mandelbrot set are both asymptotically self-similar in the point $z = c$. So, the associated limit objects $L_J$ and $L_M$ are essentially the same; they differ only by some scaling and a rotation. In the Mandelbrot set antenna there is scaling but there is no rotation. To see it, in fig. 3a we can see the neighbourdhood of the Misiurewicz point $M_{3,1} = -1.54368901269207...$ in the Mandelbrot set, and in fig. 3b we can see the neighbourdhood of the same point in the Julia set (for the parameter value $c = -1.54368901269207...$). Both figures are the same, except for an scaling. Likewise, in this figure can clearly be seen that a Misiurewicz point has no interior.

Post and Capel [32] have reported their occurrence where the two-piece chaotic attractor merges into a one-piece chaotic attractor. When one chooses alternatingly the left and the right branch in traversing the window tree downward from the period-5 CLRLL-window, one arternatingly encounters windows of the $\mathrm{CLR}\overline{\mathrm{LL}}^{k}\mathrm{L}$-family (even period) and the $\mathrm{CLR}\overline{\mathrm{LL}}^{k}$-family (odd period). These families accumulate at the right-hand respectively the left-hand side of the band merging point. Let us go back to fig. 3. The midgets are microscopic and they can not be seen with the naked eye, but their filaments are visible. In fact, we have midgets with odd period (43, 45,...) $\mathrm{CLR}\overline{\mathrm{LL}}^{20}$,

$\mathrm{CLR}\overline{\mathrm{LL}}^{21}$,... in the left hand, and midgets with even period (42, 44, 46,...) $\mathrm{CLR}\overline{\mathrm{LL}}^{19}\mathrm{L}$, $\mathrm{CLR}\overline{\mathrm{LL}}^{20}\mathrm{L}$, $\mathrm{CLR}\overline{\mathrm{LL}}^{21}\mathrm{L}$,... in the right hand.

## 3. Characteristic Misiurewicz points

### *3.1 Separators*

As we said in the introduction, we have used the Mandelbrot set antenna to study the ordering of the hyperbolic components in one-dimensional quadratic maps [4]. For such an ordering, different levels of *separators* were required. In all the cases, separators were Misiurewicz points.

Let $\mathsf{M}_{\mathrm{HC}}$ be the set of all the hyperbolic components of the real part of the Mandelbrot set defined between $c = 1/4$ and $c = -2$ (see fig. 4a). Let the Feigenbaum region *F* be the set of components of the doubling-period cascade, between $c = 1/4$ and the Feigenbaum point $c_F$. Let the chaotic region *C* be the set of components between $c_F$ and $c = -2$. As *F* and *C* are disjoint, $\mathsf{M}_{\mathrm{HC}} = F \bigcup C$, $F \bigcap C = \varnothing$. Since *F* is already well known (both component period values and component ordering) we shall analyse *C*.

The chaotic region *C* can be divided from left to right in an infinite number of chaotic bands separated by the so-called *band-merging points* or *primary separators* $\mathrm{m}_i$ [4] (see fig. 4b). Let $\mathrm{B}_i$ be the set of components of the chaotic band *i* ($i = 0,1,2,....$) between primary separators $\mathrm{m}_i$ and $\mathrm{m}_{i+1}$. As $\mathrm{B}_i$ are disjoint, $C = \bigcup_{i=0}^{\infty} \mathrm{B}_i$, $\bigcap_{i=0}^{\infty} \mathrm{B}_i = \varnothing$. All the primary separators $\mathrm{m}_i$ are Misiurewicz points as follows

$$\mathrm{m}_i = \mathrm{M}_{2^i+1,\, 2^{i-1}}, \quad i = 0,1,2,... \tag{5}$$

With eq. (5) we can know the preperiodic and period values of every band-merging point. There is only one apparent exception: $\mathrm{m}_0$. If we apply this formula for $\mathrm{m}_0$ we shall obtain $\mathrm{m}_0 = \mathrm{M}_{2,\,1/2}$, i.e. preperiod-2 and eventually period-½. But period-½, which has no physical sense, is also period-1: then it is $\mathrm{m}_0 = \mathrm{M}_{2,1}$, as it really is. We call this separator a boundary crisis separator, according to [33], because the crisis causes the termination of the attractor (bifurcation diagram) and its basin. In the Mandelbrot real map this occurs at $c = -2$, the end of the antenna of the period-1 cardioid which symbolic sequence is C.

The chaotic band $B_i$ can be divided from right to left in an infinite number of trees separated by the so-called *secondary separators* $m_{i,j}$ [4] (see fig. 4c). In fig. 5, we can see the particular case of the chaotic band $B_0$ divided from right to left in an infinite number of trees. Different partitions of the hyperbolic components can be made, but the more natural of them is that shown in fig. 5. Let us go back to fig.4c. Let $T_{i,j}$ be the set of components of the tree generated by the midget *i,j* ($j = 1,2,3,...$) of the chaotic band *i* between the secondary separators $m_{i,j}$ and $m_{i,j+1}$. As $T_{i,j}$ are disjoint, $B_i = \bigcup_{j=1}^{\infty} T_{i,j}$, $\bigcap_{j=1}^{\infty} T_{i,j} = \varnothing$. Secondary separators $m_{i,j}$ are Misiurewicz points as follows

$$m_{i,j} = M_{(j+1)2^i+1,\ 2^i}, \quad i = 0,1,2,..., \quad j = 1,2,3,... \tag{6}$$

All the separators in the chaotic band $B_i$ (secondary separators or tree separators) have a preperiod that can be calculated according to eq. (6), and a period that always is $2^i$. This is the minimum value of a Misiurewicz point period in a chaotic band for all the types of Misiurewicz points.

A double partition of $T_{i,j}$ (on the right and on the left, from the exterior to the centre) in an infinite number of tree groves can be made. A particular case, for the tree $T_{0,1}$, can be seen in fig. 6. In this case every grove has only one tree. The tree groves are separated in between by the so-called *tertiary separators* $m_{i,j,k}$ on the right and $m_{i,j,k^*}$ on the left [4] (see fig. 4d). Let $G_{i,j,k}$ be the set of components of the grove $k$ ($k = 1,2,3,...$) belonging to the right part of the tree $T_{i,j}$, between the tertiary separators $m_{i,j,k}$ and $m_{i,j,k+1}$, and let $G_{i,j,k*}$ be the set of components of the grove $k*$ ($k* = 1,2,3,...$) belonging to the left part of the tree $T_{i,j}$, between the tertiary separators $m_{i,j,k*}$ and $m_{i,j,k*+1}$. Since both $G_{i,j,k}$ and $G_{i,j,k^*}$ are disjoint, $T_{i,j}=\bigcup_{k=1}^{\infty}G_{i,j,k}\bigcup_{k^*=1}^{\infty}G_{i,j,k^*}$, $\bigcap_{k=1}^{\infty}G_{i,j,k}\bigcap_{k^*=1}^{\infty}G_{i,j,k^*}=\varnothing$. Tertiary separators $m_{i,j,k}$ (or $m_{i,j,k^*}$) are Misiurewicz points as follows

$$m_{i,j,k}=M_{[k(j+2)-1]2^i+1,\ 2^i},\quad i=0,1,2,...,\quad j=1,2,3,...,\quad k=1,2,3,... \tag{7}$$

The grove $G_{i,j,k}$ is generated by the midgets $\left[(j+2)k+v+2\right]2^i$ with $v=0,1,...,j-1$. Therefore, the grove is constituted at most by a disjoint union of *j* trees. A particular case for the tree $T_{0,3}$ can be seen in fig. 7 where, due to $j=3$, every grove has 3 trees. These *j* trees are separated by the so-called *tertiary subseparators* $m_{i,j,k}^{v}$. Tertiary subseparators $m_{i,j,k}^{v}$ are Misiurewicz points as follows

$$m_{i,j,k}^{v}=M_{[k(j+2)+(v+2)-1]2^i+1,\ 2^i},\quad i=0,1,2,...,\quad j=1,2,3,...,\quad k=1,2,3,...,\quad v=1,2,...,j-1 \tag{8}$$

We could continue with quaternary, quinary, ... separators but the process would be the same: in all the cases we would obtain a Misiurewicz point, i.e., all the separators used in the Mandelbrot real map partitions are Misiurewicz points. All the Misiurewicz points treated so far have a common feature: their period is $2^i$. But this is the value of the period of the chaotic band where they are. Therefore, in the period-$2^i$ chaotic band, the period of all the Misiurewicz points of this type is $2^i$. These points are called *characteristic Misiurewicz points* because their period is the same as the period of the chaotic band where they are. Next we shall see some properties of these points.

*3.2. Properties of the characteristic Misiurewicz points*

We shall begin by seeing some properties of the period and preperiod of these points.

a) As we have just seen, the period of any characteristic Misiurewicz point in the period-$2^i$ chaotic band $B_i$ is $p=2^i$.

b) As can experimentally be seen, the lower preperiod of a characteristic Misiurewicz point in the period-$2^i$ chaotic band $B_i$ is $3\cdot 2^i+1$ (without taking into account the merging points or limits of the band) or $2^{i+1}+1$ (if we take into account the limits of the band). From this value, the preperiod increases $2^i$ by $2^i$.

Taking into account properties a) and b), we have that in the period-$2^i$ chaotic band $B_i$, all the characteristic Misiurewicz points (without the limits of the band) have the form $M_{(3+q)2^i+1,\ 2^i}$, where $i = 1,2,3,...$ and $q = 0,1,2,...$ When $q = 0$, the minimum preperiod value is obtained. So, the lower value in the chaotic band $B_0$ is $M_{4,1}$ followed by $M_{5,1}$, $M_{6,1}$,... ; the lower value in the chaotic band $B_1$ is $M_{7,2}$ followed by $M_{9,2}$, $M_{11,2}$,...; the lower value in the chaotic band $B_2$ is $M_{13,4}$, followed by $M_{17,4}$, $M_{21,4}$,... and so on.

*3.3. Graphic and symbolic analysis of the characteristic Misiurewicz points*

*3.3.1. External branches emerge from the Misiurewicz points*

As can be seen in fig. 5 and as we previously said, characteristic Misiurewicz points separate external branches of two consecutive bands, trees or groves. We can easily calculate the patterns of the superstable orbits of these external branches starting from the pattern of the unstable orbit of the Misiurewicz point given. In fig. 8a we have represented four Misiurewicz points of the period-1 chaotic band $B_0$ that are from right to left: a secondary separator or tree separator $m_{0,3} = M_{5,1}$, two tertiary subseparators $m^1_{0,3,1} = M_{8,1}$ and $m^2_{0,3,1} = M_{9,1}$, and a tertiary separator or grove separator $m_{0,3,2} = M_{10,1}$.

Let us note $M_{5,1}$ at the right of fig. 8a. Numbers show the period of the components in the branches. The pattern of this Misiurewicz point is $(CLR^3)L$. Starting from this pattern we can write the sequence $P_6 = CLR^3L$, $P_7 = CLR^3L^2$, $P_8 = CLR^3L^3$, $P_9 = CLR^3L^4$,... that tends to this Misiurewicz point. According with the ordering rule of the symbolic sequences seen in the paragraph 2.5., we have $P_6 \lhd P_7$, $P_7 \rhd P_8$, $P_8 \lhd P_9$,..., where the symbols "$\lhd$" and "$\rhd$" must be read as "is dominated by" (or "precedes", according with the Sharkovsky ordering) and "dominates" [20]. Therefore, the route $P_6$, $P_7$, $P_8$, $P_9$,.... is a *zigzag route* and we can reach the Misiurewicz point starting from the component of period 6 and jumping from one branch to the other one, as indicated in the thin lines of the figure, until eventually reach the Misiurewicz point at the infinity. Let us note that the external branches of the consecutive trees that lead to $M_{5,1}$ are $P_6$, $P_8$, $P_{10}$,... (right branch, formed by dominated points) and $P_7$, $P_9$, $P_{11}$,... (left branch, formed by dominant points).

In the four cases of fig. 8a, the period increases one unit in each jump, which coincides with the period of the chaotic band where they are, which is one. However, in the three cases of fig. 8b, which represent Misiurewicz points in $B_1$, the period of the components increases two units in each jump, which coincides with the period of the chaotic band where they are, which is two. Let us use now $m_{1,2} = M_{7,2}$ of the period-2 chaotic band given at the right of fig. 8b. The pattern of this Misiurewicz point is $(CLRL^4)LR$. Starting from this pattern we can write the even period sequence (it is not possible to find an odd period sequence in $B_1$) $P_{10} = CLRL^5RL$, $P_{12} = CLRL^5RLRL$, $P_{14} = CLRL^5RLRLRL$,... that tends to this Misiurewicz point. According with the ordering rule of the symbolic sequences seen in the paragraph 2.5., we have $P_{10} \lhd P_{12}$, $P_{12} \rhd P_{14}$, $P_{14} \lhd P_{16}$,.... Again the route $P_{10}$, $P_{12}$, $P_{14}$, $P_{16}$, .... is a *zigzag route* as in the case of the characteristic Misiurewicz point of the band $B_0$. The external branches of the consecutive trees that lead to $M_{7,2}$ are $P_{10}$, $P_{14}$, $P_{18}$,... (right branch, formed by dominated points) and $P_{12}$, $P_{16}$, $P_{20}$,... (left branch, formed by dominant points).

The same behaviour is observed in following bands. Therefore, we can enunciate the following property: *When an hyperbolic component jumps to another one according to what has been described before in order to reach to a characteristic Misiurewicz point in the limit, the period increases in the same value as the period of the chaotic band where they are.*

### *3.3.2. A Misiurewicz point is the common limit of consecutive external branches*

A Misiurewicz point not only separate external branches of two consecutive bands, trees or groves, but is the common limit of both consecutive branches. It is easy to calculate the pattern of a Misiurewicz point starting from the symbolic sequences of the hyperbolic components that form the external branches whose limit is that Misiurewicz point. Let us note $M_{5,1}$ again, given at the right of fig. 8a. The sequence of patterns we have to cover to go from the component of period 6, which is $CLR^3L$, to $M_{5,1}$ is: $CLR^3L$, $CLR^3LL$, $CLR^3LLL$,... As can be seen, from a certain value letters recur, or, in other words, are periodic. In such a case, the symbolic sequence of the Misiurewicz point is easily deduced from this sequence: *The preperiod is the initial not repeated part and the period is the repeated part*. Therefore, according to the notation we introduced before, $M_{5,1} = (CLR^3)L$. We have obtained period one because we are in the period-1 chaotic band, according to what we saw before. Let us use now $M_{7,2}$ of the period-2 chaotic band given at the right of fig. 8b. The sequence of patterns is: $CLRL^5RL$, $CLRL^5RLRL$, $CLRL^5RLRLRL$,... Let us note that here we reach the period with LR, then $M_{7,2} = (CLRL^4)LR$.

Let us note the power of this method which allows us to deduce the symbolic sequence of a Misiurewicz point and in addition its preperiod and its period if we know the symbolic sequences of the components of the branches.

There are other different Misiurewicz points whose period is not the same as the chaotic band period. These points will be treated in the following paragraph.

## 4. Non-characteristic Misiurewicz points

We shall call the Misiurewicz points whose periods are not the same as those of the chaotic band where they are located *non-characteristic Misiurewicz points*. Next let us see some properties of the period and preperiod of these points.

a) The period of a non-characteristic Misiurewicz point which is in the period-$2^i$ chaotic band $B_i$ is $r \cdot 2^i$ where $r = 2,3,4,...$ Therefore, the lower value of the period of a non-characteristic Misiurewicz point is $2^{i+1}$.

b) The lower preperiod of a non-characteristic Misiurewicz point in the period-$2^i$ chaotic band $B_i$ is $2^{i+1}+1$, the same as in the case of characteristic ones when limits were included, and increases $2^i$ by $2^i$.

By taking into account properties a) and b), we have that in the period-$2^i$ chaotic band $B_i$ (without taking into account limits), all the non-characteristic Misiurewicz points have the form: $M_{2^{i+1}+2^i\cdot q+1,\ 2^i\cdot r}$, where $i=0,1,2,...$, $q=0,1,2,...$ and $r=2,3,4,...$ When $q=0$ and $r=2$ the minimum preperiod of each $B_i$ is obtained. So, in the period-1 chaotic band $B_0$ the non-characteristic Misiurewicz point with a lower preperiod and period is $M_{3,2}$, in $B_1$ is $M_{5,4}$, in $B_2$ is $M_{9,8}$, and so on. All the others are obtained by increasing the preperiod, the period or both $2^i$ by $2^i$.

In the last paragraph we saw that Misiurewicz points were separators (of bands, trees, or groves). Do non-characteristic Misiurewicz points have the same nature? As we can see in Table I, there are three points $M_{4,3}$, which are $M_{4,3}^{(1)}=(CLR^2)LRL$, $M_{4,3}^{(2)}=(CLR^2)RL^2$ and $M_{4,3}^{(3)}=(CLR^2)R^2L$ [26]. In fig. 9 we give a sketch of these three points, all of them with period three though they are in the period-1 chaotic band.

Let us consider the first point (see fig. 9a). The sequence $P_7=CLR^2LRL$, $P_{10}=CLR^2LRL^2RL$, $P_{13}=CLR^2LRL^2RL^2RL,...$ tends to $M_{4,3}^{(1)}$. By ordering we obtain $P_7 \triangleright P_{10} \triangleright P_{13}...$ Therefore, $P_7$, $P_{10}$, $P_{13}$, ... is a non-zigzag route, and all of them are on the left of the Misiurewicz point. But this is not the only possible route since a cyclic permutation of the period letters allows to find two more which are defined by the sequences $P_8=CLR^2LRL^2$, $P_{11}=CLR^2LRL^2RL^2$, $P_{14}=CLR^2LRL^2RL^2RL^2,...$ and $P_9=CLR^2LRL^2R$, $P_{12}=CLR^2LRL^2RL^2R$, $P_{15}=CLR^2LRL^2RL^2RL^2R,...$ giving place to the non-zigzag routes $P_8$, $P_{11}$, $P_{14}$, ... and $P_9$, $P_{12}$, $P_{15}$,... No one of the three possible routes that lead to the Misiurewicz point $M_{4,3}^{(1)}$ coincide with any external branch of some of the trees previously seen in the ordering of the fig. 5. It is the same with the routes that lead to the other two points $M_{4,3}^{(2)}$ and $M_{4,3}^{(3)}$ (in fig. 9b we have three non-zigzag routes and in fig. 9c we have three zigzag routes). Therefore, we never find any point of this type when we manage our ordering [4]. However, these points are also separators of external branches of consecutive trees. To explain it, we give fig. 10. In the lower part of this figure, we represent a portion of the antenna ($-1.676 < c < -1.648$) in the neighbourhood of the non-characteristic Misiurewicz point $\mathrm{M}_{4,2}^{(1)} = (\mathrm{CLRL})\mathrm{LR}$ and, in the upper part, a sketch with the hyperbolic components ordered from top to bottom when their period increases. The natural partition in trees always causes separators of period one in the period-1 chaotic band. However, if we reorganise the ordering in a forced way as is shown in the dashed line of the figure, a separator $\mathrm{M}_{4,2}$, which seems hidden at first, can be obtained. If we force the reorganising more and more, separators with greater periods can be obtained.

*4.1 The end of the antenna of a primary midget*

In accordance with Yorke et al [34] a primary window is one which is not a window within a window in the bifurcation diagram. For example, the period $3 \times 3 = 9$ window within the period-3 window is not a primary window. Perhaps the most widely shown example of a primary window is the period-3 window. If we work with the antenna of the Mandelbrot set instead of the Mandelbrot real map bifurcation diagram, we have midgets instead of windows, and the end of a window is the end of an antenna. Grebogi et al [33] have shown that the parameter value $c = 1.790327492...$ is an interior crisis point, the end of the period-3 primary window, that generates an unstable orbit in the map $x_{n+1} = c - x_n^2$.

We have seen that the end of the antenna of a primary midget of the Mandelbrot set antenna is a non-characteristic Misiurewicz point. For example, the end of the antenna of period-3 midget CLR is $\mathrm{M}_{4,3}^{(1)} = (\mathrm{CLR}^2)\mathrm{LRL}$ for $c = -1.79032749199934...$, and the end of the antenna of period-5 midget $\mathrm{CLR}^2\mathrm{L}$ is $\mathrm{M}_{6,5}^{(9)} = (\mathrm{CLR}^2\mathrm{L}^2)\mathrm{LR}^2\mathrm{LR}$ for $c = -1.862331090....$

## 5. Conclusions

The set of exceptional points which is constituted by the Misiurewicz points has been treated here from a more experimental physical point of view than those used in the habitual mathematical works about this topic. The nature of separators of these points has been emphasised. We have shown that they are a suitable tool to classify hyperbolic components of the one-dimensional quadratic maps.

The notion of characteristic and non-characteristic Misiurewicz points has been introduced. The first ones are those that have the same period as the chaotic band where they are located and the second ones are those that have a different period from the chaotic band where they are located.

The preperiod and the period of each one of the separators have been calculated and formulas which determine the possible preperiods and periods in all the cases have been given. Likewise, the minimum possible values have been calculated.

The symbolic sequence corresponding to each of the Misiurewicz points has been deduced. A graphic tool to determine the symbolic sequence, the preperiod and the period of a Misiurewicz point has been introduced starting from the external branches of the two consecutive trees that delimit the point. Likewise, starting from the pattern of a Misiurewicz point, the routes (zigzag or non-zigzag) through hyperbolic components to reach the Misiurewicz point are deduced.

We have tried to give a more experimental physical and less conceptual view of the Misiurewicz points. They have been quantified and ordered for the case of one-dimensional quadratic maps, and so we have hopefully made them more accessible and familiar to us.

**Acknowledgements**

This research was supported by the "Programa Nacional" under grant nº TIC95-0080 (project: "Sistema criptográfico de protección de datos para red digital de servicios integrados RDSI").

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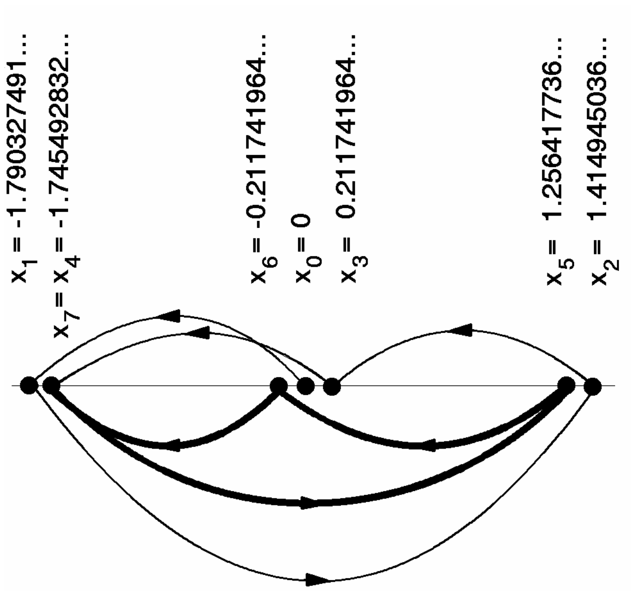


Fig. 1. Sketch of a $M_{4,3}$ (preperiod-4 and eventually period-3 Misiurewicz point) for the map $f(x) = x^2 + c$ when $c = -1.79032749199934....$

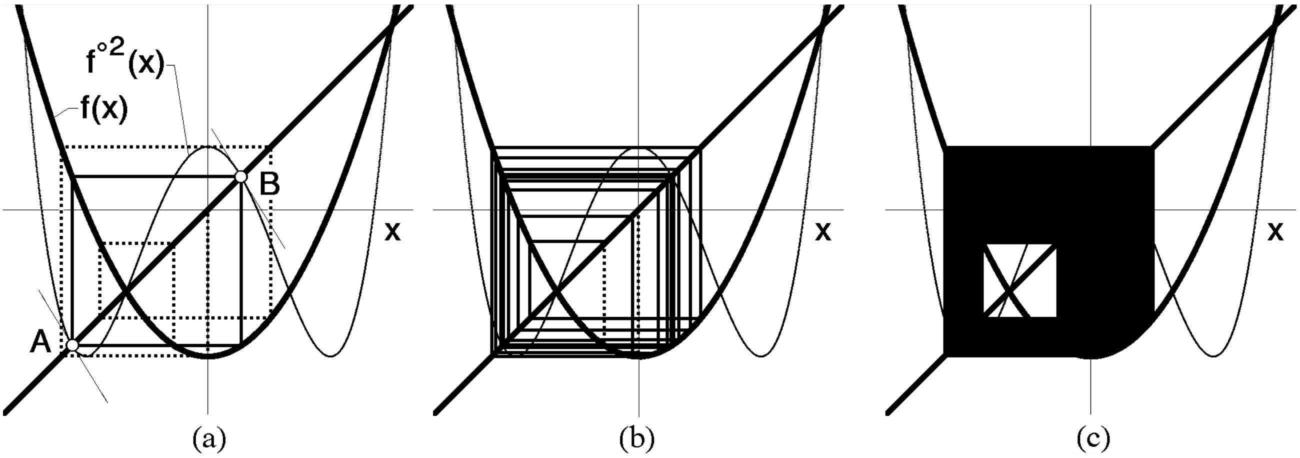


Fig. 2. Graphical iteration of the Misiurewicz point $M_{5,2}^{(1)}$ for the map $f(x) = x^2 + c$ when $c = -1.43035763245130...$

(a) 20 iterations. (b) 120 iterations. (c) 5000 iterations.

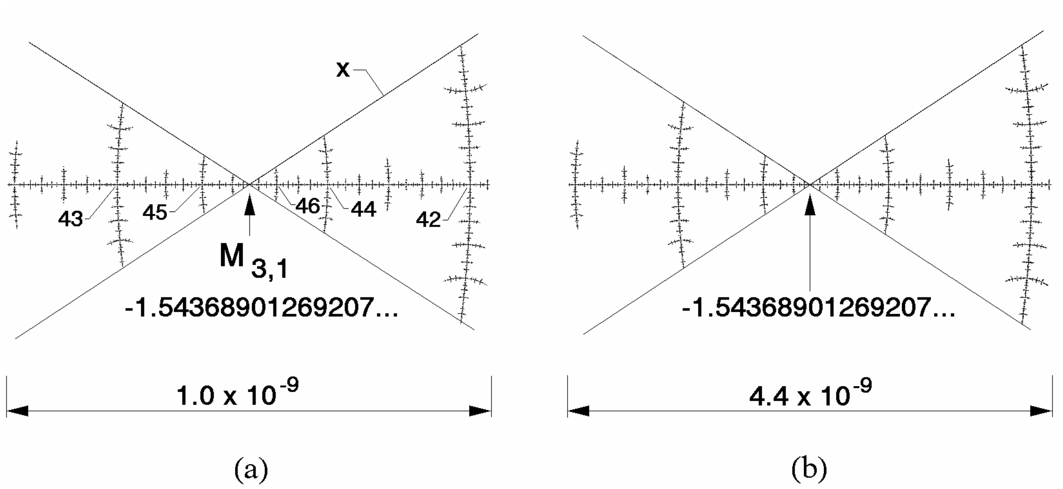


Fig. 3. Graphical view of Tan Lei theorem in the Mandelbrot set antenna. The X form straight lines do not belong to the sets: they serve to show the asymptotic behaviour of the periodic components.

(a) Mandelbrot set in the neighbourdhood of $M_{3,1}$.

(b) Julia set (for the parameter value of $M_{3,1}$) in the same neighbourdhood.

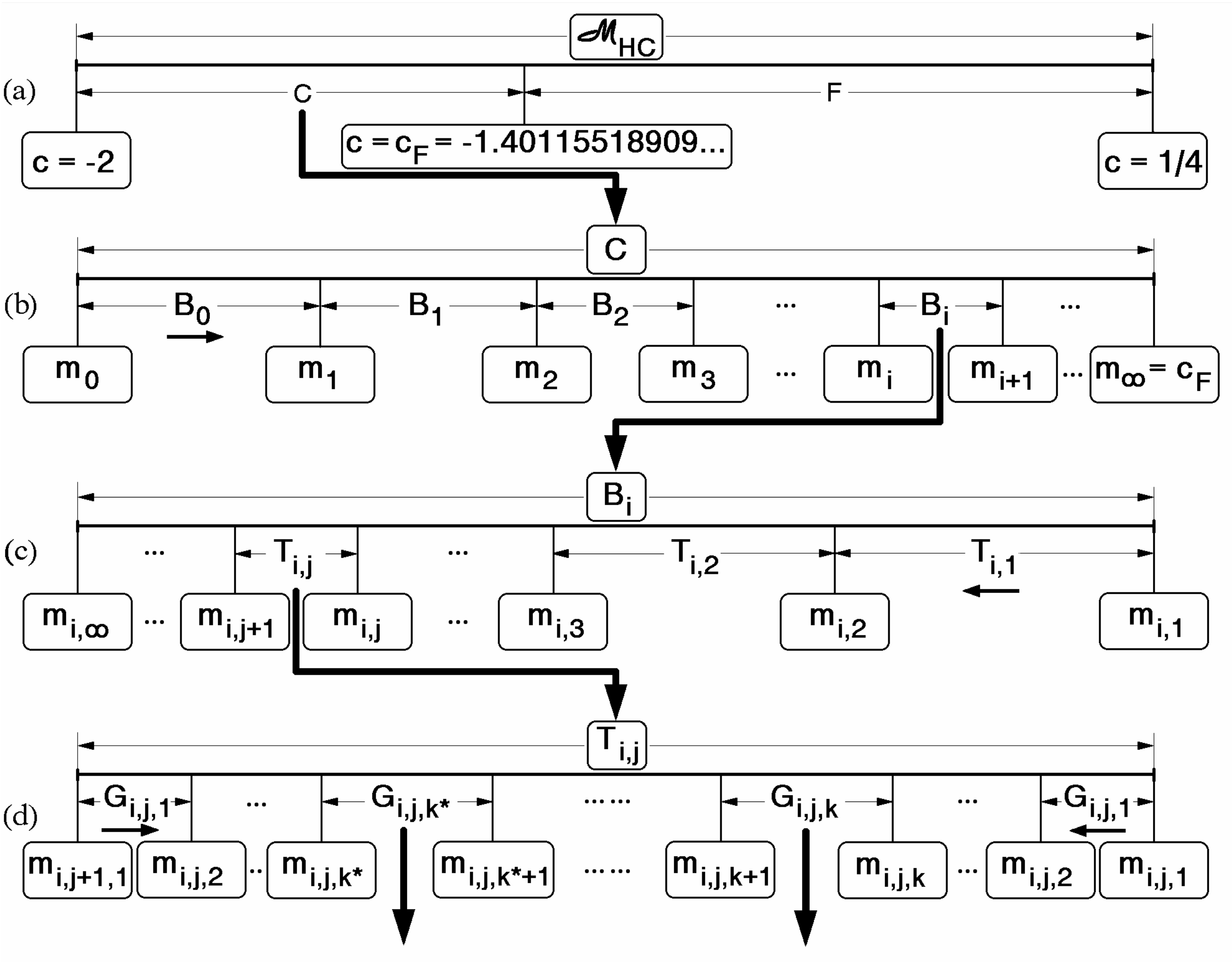


Fig. 4. Sketch of the hyperbolic component set partition of the Mandelbrot real map. (a) Partition of $M_{HC}$. (b) Partition of C. (c) Partition of $B_i$. (d) Partition of $T_{i,j}$.

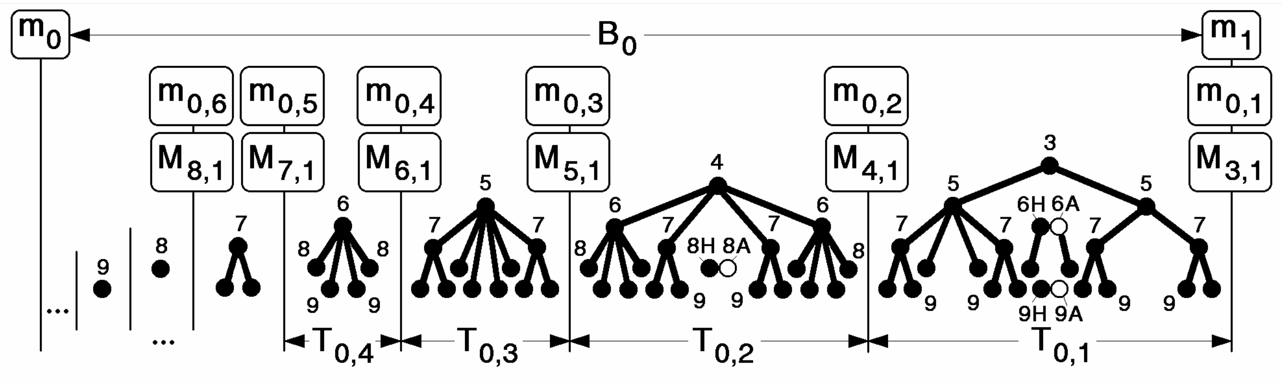


Fig. 5. Hyperbolic components of periods less or equal to 9 of the chaotic band $B_0$ are represented. The greater the period, the lower the component. A partition of the chaotic band $B_0$ in an infinite number of trees by means of separators (which are Misiurewicz points) is shown.

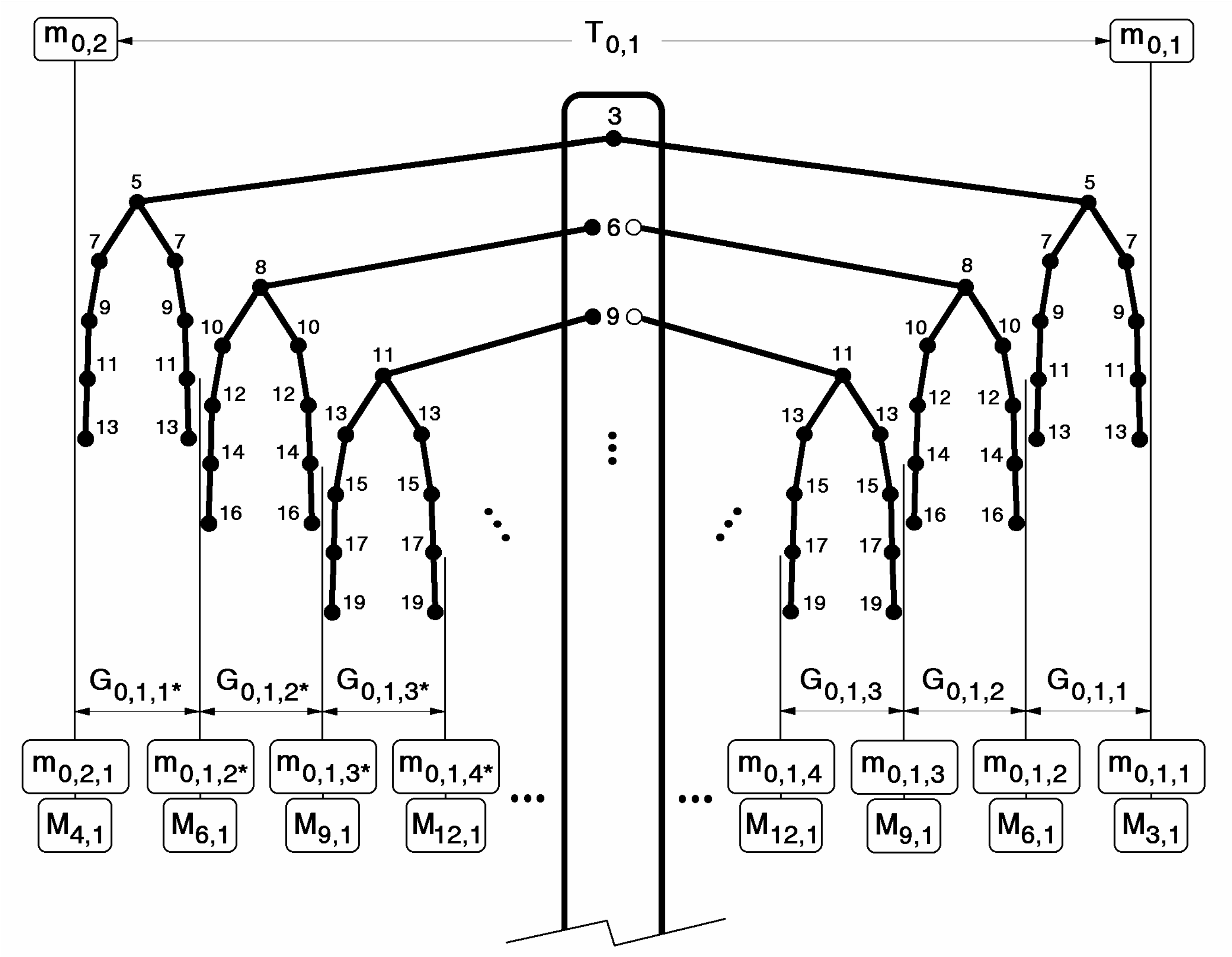


Fig. 6. The tree $T_{0,1}$. Only external branches of the new trees generated by the Fourier-harmonics and the Fourier-antiharmonics of the period-3 midget are drawn.

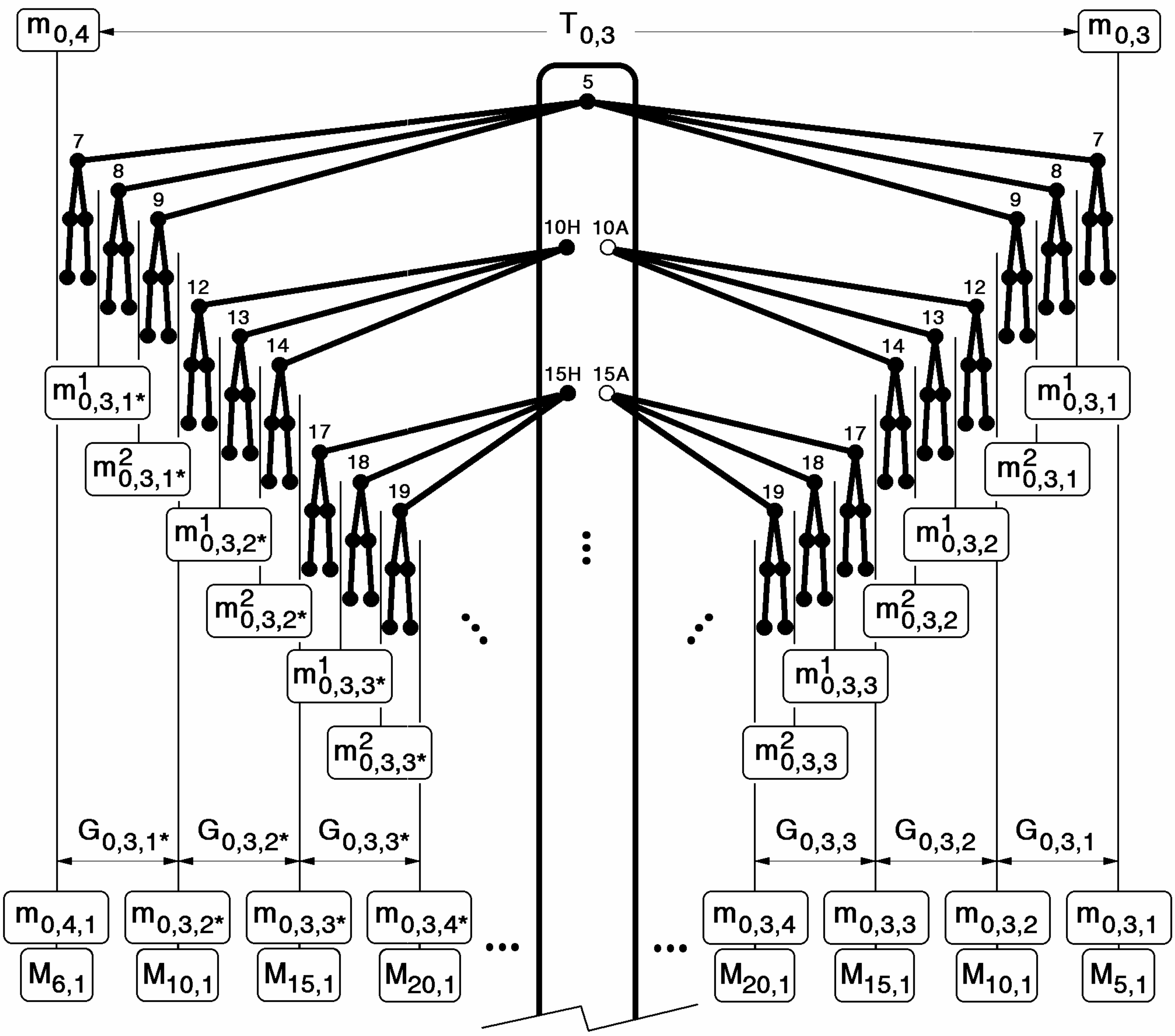


Fig. 7. A sketch of the tree $T_{0,3}$ where secondary and tertiary separators, and tertiary subseparators are shown.

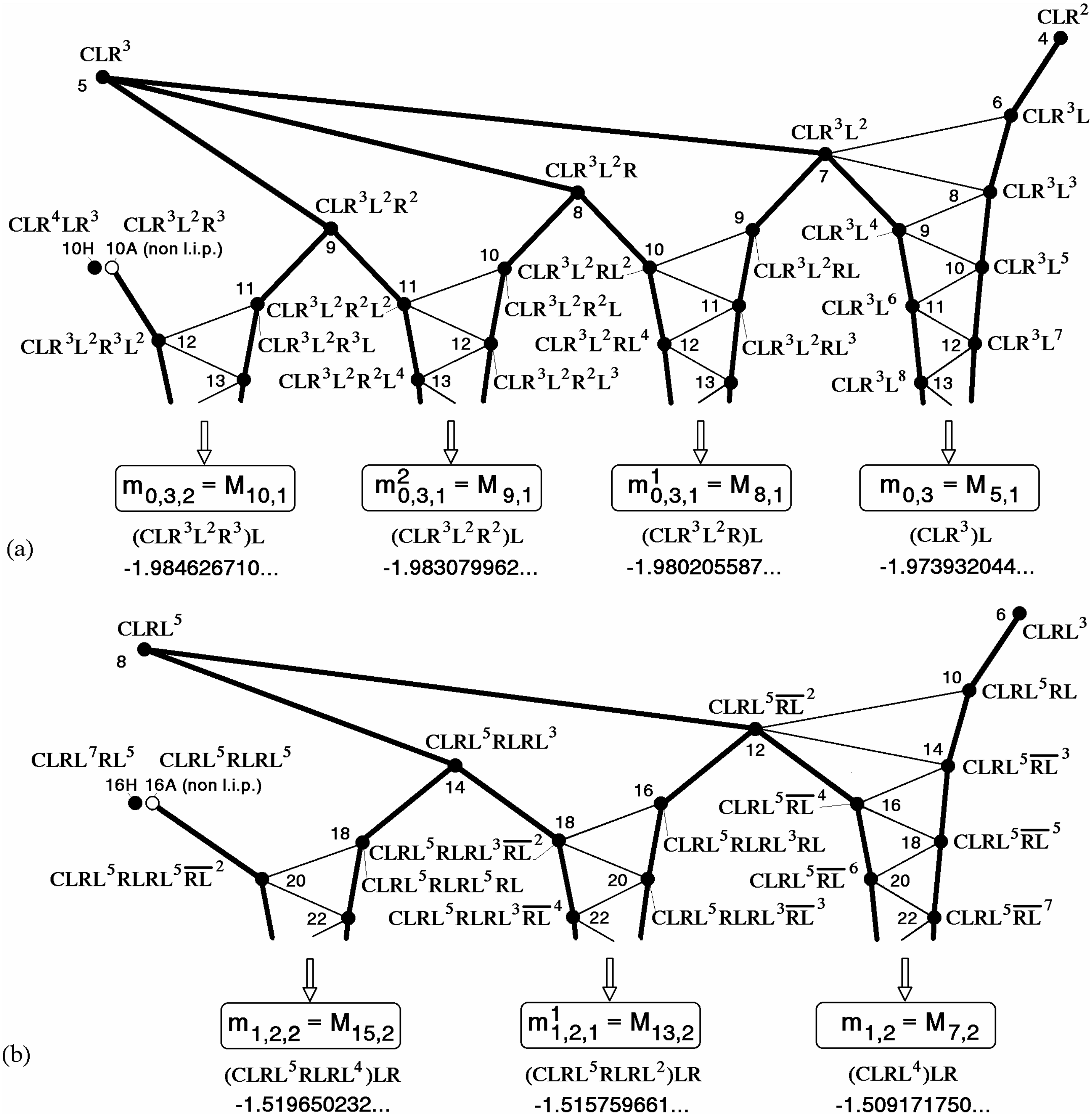


Fig. 8. Sketches of characteristic Misiurewicz points for the map $f(x) = x^2 + c$.

(a) Misiurewicz points in the period-1 chaotic band.

(b) Misiurewicz points in the period-2 chaotic band.

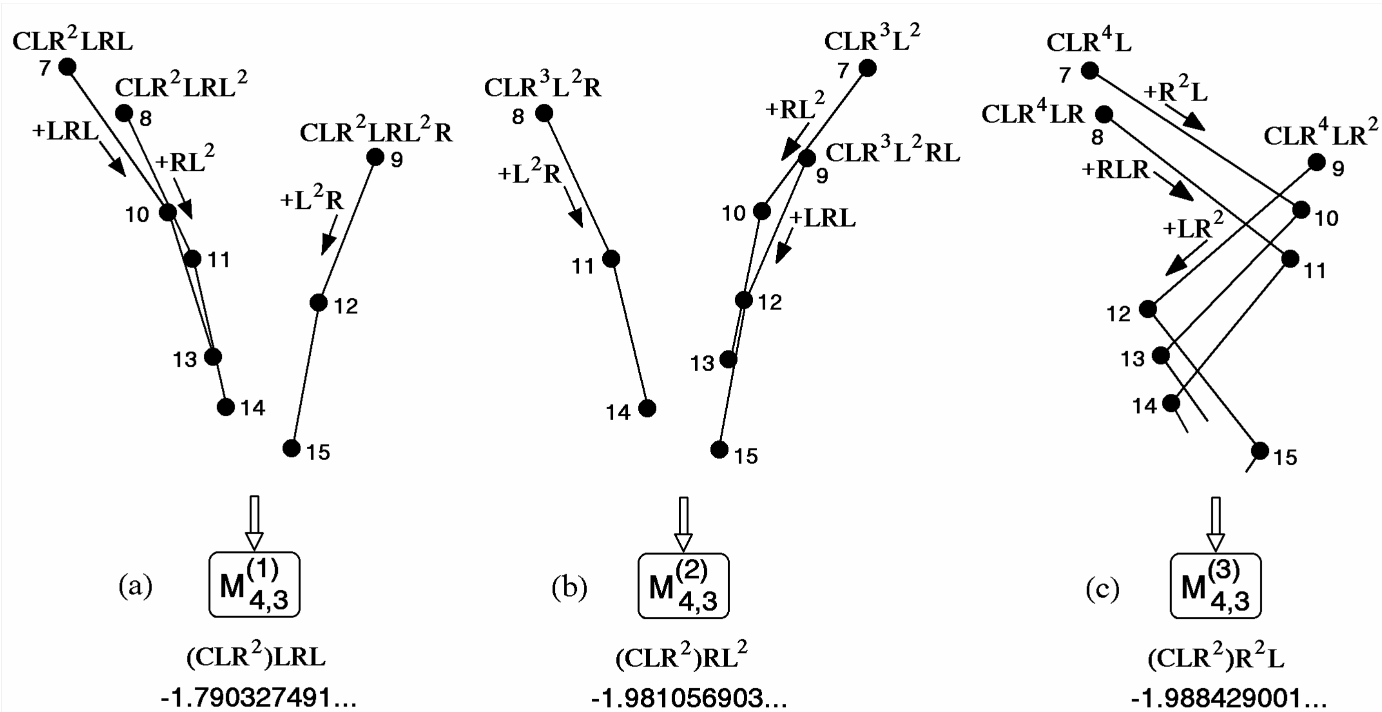


Fig. 9. Sketches of the non-characteristic Misiurewicz points $M_{4,3}$ in the period-1 chaotic band for the map $f(x) = x^2 + c$.

a) $M^{(1)}_{4,3}$, b) $M^{(2)}_{4,3}$, c) $M^{(3)}_{4,3}$ .

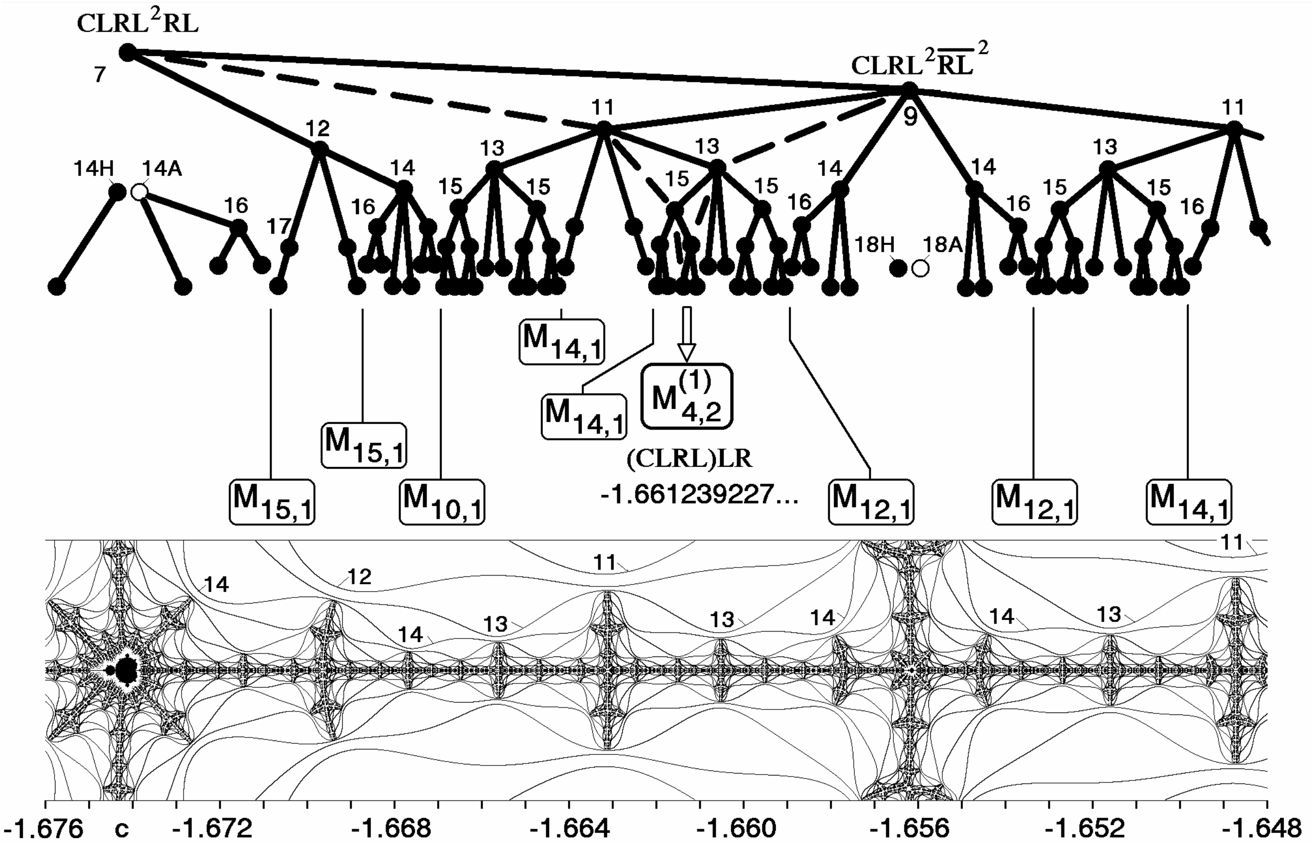


Fig. 10. A sketch to show how the non-characteristic Misiurewicz point $M_{4,2}^{(1)}$ can be graphically obtained. In the lower part, a portion of the antenna of the Mandelbrot set $(-1.676 < c < -1.648)$ is shown. In the upper part, a sketch of hyperbolic components (the greater the period the lower the component) is shown, and some characteristic Misiurewicz points are also drawn.